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ON THE VIBRATIONS OF A SEMI-INFINITE BEAM WITH INTERNAL AND EXTERNAL FRICTION*

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The spectrum of a problem associated with the vibrations of a semi-infinite beam with internal (Voigt material) and external viscous friction is investigated. Under certain conditions a domain is defined in which a complex discrete spectrum is possible. Sufficient conditions are obtained for there to be no complex discrete spectrum.

Consider the equation

$$\alpha\frac{\partial^5 u}{\partial\tau\partial^4 x}+\frac{\partial^4 u}{\partial x^4}+\frac{\partial}{\partial x}g(x)\frac{\partial u}{\partial x}+k(x)\frac{\partial u}{\partial\tau}+\frac{\partial^2 u}{\partial\tau^2}=0 \tag{1}$$

$$\alpha=\nu_1\sqrt{\frac{EJ}{m}}>0,\quad k(x)=\frac{\nu_2(x)}{\sqrt{mEJ}}\geqslant 0,\quad g(x)=\frac{P(x)}{EJ}\,\tau=\sqrt{\frac{EJ}{m}}\,t$$

that describes the vibrations of an elastic beam possessing internal friction (a Voigt material) /1/ and external friction. Here m is the mass per unit length, $E\nu_1$ is the coefficient of internal friction, $E\nu_2(x)$ is the coefficient of external friction, EJ is the bending stiffness, t is the time, and $P(x)$ is a distributed tensile (compression) force.

The boundary conditions corresponding to a rigidly clamped left end of the rod have the form

$$u(0,\tau)=du(x,\tau)/dx\,|_{x=0}=0 \tag{2}$$

After substituting $u(x,\tau)=e^{\lambda\tau}y(\lambda,x)$ into (1) and (2), we obtain a problem on the half-axis

$$y^{IV}+\frac{[g(x)y']'}{1+\alpha\lambda}+\frac{\lambda k(x)y}{1+\alpha\lambda}+\frac{\lambda^2 y}{1+\alpha\lambda}=0 \tag{3}$$

$$y(\lambda,0)=y'(\lambda,0)=0 \tag{4}$$

We will assume in the sequel that the following conditions are satisfied:

1) $k(x)$ is a continuously differentiable function and

$$k(x)\geqslant 0,\quad \int_0^\infty k(x)\exp(\varepsilon x^{1+\delta})\,dx<\infty,\quad \varepsilon>0,\quad \delta>0$$

2) $g(x)$ is a continuously differentiable function and

$$|g(x)|\leqslant g_{\max}<\infty,\quad \int_0^\infty |g(x)|\exp(\varepsilon x^{1+\delta})\,dx<\infty,$$

$$\int_0^\infty |g'(x)|\exp(\varepsilon x^{1+\delta})\,dx<\infty$$

Let I be the interval $(-\infty,-\alpha^{-1})$, and let O a circle of radius $r_1=\alpha^{-1}$ with centre at the point $\lambda=-\alpha^{-1}$. Obviously the quantity $\lambda^2/(1+\alpha\lambda)$ is real and non-negative on I and O.

Lemma 1. If conditions 1) and 2) are satisfied, then for $\lambda\neq 0$, $\lambda\neq-\alpha^{-1}$ there exists a four-valued function $y(\lambda,x)$ that is a solution of (3) and satisfies the condition

$$\lim_{x\to\infty}y(\lambda,x)e^{\beta x}=1,\quad \beta=\left(\frac{-\lambda^2}{1+\alpha\lambda}\right)^{1/4} \tag{5}$$

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This function and its derivative with respect to x are analytic in λ for fixed x on a four-sheeted Riemann surface (corresponding to $\beta(\lambda)$) with the branch points $\lambda = 0$ and $\lambda = -\alpha^{-1}$.

Proof. Eq.(3) with boundary condition (5) is equivalent to the integral equation

$$y(\lambda, x) = -\int_x^\infty \frac{\sin\beta(x-t)\,dt}{\beta}\int_t^\infty \frac{\operatorname{sh}\beta(t-s)}{\beta}\times \left\{\frac{[g(s)\,y'(\lambda, s)]'}{1+\alpha\lambda}+\frac{\lambda k(s)\,y(\lambda, s)}{1+\alpha\lambda}\right\}ds \tag{6}$$

Integrating by parts we obtain for $\omega(\lambda, x) = y(\lambda, x)\,e^{\beta x}$ the integral equation

$$\begin{gathered}\omega(\lambda, x) = 1 + \Omega(\lambda, \omega(\lambda, s)) \\ \Omega(\lambda, \omega(\lambda, s)) = -\lambda\int_x^\infty \frac{\sin\beta(x-t)\,dt}{\beta^2(1+\alpha\lambda)}\int_t^\infty \operatorname{sh}\beta(t-s)\,e^{\beta(x-s)}k(s)\,\omega(\lambda, s)\,ds + \\ \int_x^\infty \frac{e^{\beta(x-t)}\sin\beta(x-t)}{\beta(1+\alpha\lambda)}g(t)\,\omega(\lambda, t)\,dt + \int_x^\infty \frac{\sin\beta(x-t)\,dt}{\beta(1+\alpha\lambda)}\times \\ \int_t^\infty \operatorname{ch}\beta(t-s)\,e^{\beta(x-s)}g'(s)\,\omega(\lambda, s)\,ds - \\ \int_x^\infty \frac{\sin\beta(x-t)\,dt}{(1+\alpha\lambda)}\int_t^\infty \operatorname{sh}\beta(t-s)\,e^{\beta(x-s)}g(s)\,\omega(\lambda, s)\,ds\end{gathered} \tag{7}$$

We will seek the solution in the form of the series

$$\begin{gathered}\omega(\lambda, x) = \omega_0(\lambda, x) + \omega_1(\lambda, x) + \ldots \\ \omega_0(\lambda, x) = 1, \quad \omega_{n+1}(\lambda, x) = \Omega(\lambda, \omega_n(\lambda, s))\end{gathered} \tag{8}$$

We will use the obvious rough estimates

$$|e^{\beta(x-s)}\sin\beta(x-t)\operatorname{sh}\beta(t-s)| \leqslant e^{2|\beta|s}, \quad |e^{\beta(x-t)}\sin\beta(x-t)| \leqslant e^{2|\beta|t}, \quad |e^{\beta(x-s)}\sin\beta(x-t)\operatorname{ch}\beta(t-s)| \leqslant e^{2|\beta|s}$$

Then it follows from (8) that

$$\begin{gathered}|\omega_{n+1}(\lambda, x)| \leqslant \left|\frac{\lambda}{\beta^2(1+\alpha\lambda)}\right|\left|\int_x^\infty dt\int_t^\infty d\,s e^{2|\beta|s}k(s)\,|\omega_n(\lambda, s)|\,ds + \right. \\ \frac{1}{|\beta(1+\alpha\lambda)|}\int_x^\infty e^{2|\beta|t}|g(t)|\,|\omega_n(\lambda, t)|\,dt + \frac{1}{|\beta(1+\alpha\lambda)|}\int_x^\infty dt\int_t^\infty e^{2|\beta|s}\times \\ |g'(s)|\,|\omega_n(\lambda, s)|\,ds + \frac{1}{|1+\alpha\lambda|}\int_x^\infty dt\int_t^\infty e^{2|\beta|s}|g(s)|\,|\omega_n(\lambda, s)|\,ds\end{gathered}$$

Integrating by parts in the first, third and fourth terms on the right-hand side of the last inequality, we obtain

$$\begin{gathered}|\omega_{n+1}(\lambda, x)| \leqslant \int_x^\infty F(\lambda, t)\,|\omega_n(\lambda, t)|\,dt \\ F(\lambda, t) = \left|\frac{\lambda}{\beta^2(1+\alpha\lambda)}\right|e^{2|\beta|t}k(t)\,t + \\ \left|\frac{1}{1+\alpha\lambda}\right|e^{2|\beta|t}|g(t)|\left(t+\frac{1}{|\beta|}\right) + \left|\frac{1}{\beta(1+\alpha\lambda)}\right|e^{2|\beta|t}|g'(t)|\,t\end{gathered}$$

It hence follows that

$$\begin{gathered}|\omega(\lambda, x)| \leqslant G(\lambda, x), \quad |\omega(\lambda, x) - 1| \leqslant G(\lambda, x) - 1 \\ G(\lambda, x) = \exp\int_x^\infty F(\lambda, t)\,dt\end{gathered} \tag{9}$$

i.e., the assertion of the lemma holds for the function $y(\lambda, x)$.

An analogous investigation of the equation obtained by differentiating (6) with respect to x results in the conclusion of the lemma for $y'(\lambda, x)$.

For values of λ not belonging to the interval I and the circle O, $\beta(\lambda)$ obviously has four values, two of which lie in the right-hand half-plane. Therefore, two of the branches of the function $y(\lambda, x)$ are square-integrable with respect to x for such λ. Let $y_1(\lambda, x)$ and $y_2(\lambda, x)$ be branches of the function $y(\lambda, x)$ satisfying the conditions

$$\lim_{x\to\infty} y_1(\lambda, x)e^{|\beta|x} = 1, \quad \lim_{x\to\infty} y_2(\lambda, x)e^{i|\beta|x} = 1 \tag{10}$$

for λ within the interval $J = (-\infty, -2\alpha^{-1})$ and the upper semicircle of O. The analytic continuations of these functions into a domain exterior to the circle O with a slit from $-\infty$ to $-2\alpha^{-1}$ are square-integrable with respect to x.

We introduce the solution of (3)

$$\psi_1(\lambda, x) = y_1'(\lambda, 0)y_2(\lambda, x) - y_2'(\lambda, 0)\, y_1(\lambda, x)$$

The zeros of the function $\psi_1(\lambda, 0)$ lying outside the circle O but not in the interval J correspond to the eigenvalues of problem (3) and (4). Because of the analyticity of $\psi_1(\lambda, 0)$ they form here a discrete set that has no condensation points in a finite domain.

Let us define the function

$$\psi_2(\lambda, x) = y_1'(\lambda, 0)\, y_3(\lambda, x) - y_3'(\lambda, 0)y_1(\lambda, x)$$

where $y_3(\lambda, x)$ is the solution of (3) that satisfies the condition

$$\lim_{x\to\infty} y_3(\lambda, x)e^{-i|\beta|x} = 1$$

for λ lying on the upper semicircle of O. The zeros of this function lying within the circle O but not on the segment $[-2\alpha^{-1}, -\alpha^{-1}]$ correspond to eigenvalues. The discrete set of these eigenvalues can have only $\lambda = -\alpha^{-1}$ the condensation point of zeros, as follows from the possibility of analytic continuation of the function $\psi_2(\lambda, 0)$.

Lemma 2. A continuous spectrum lies on the circle O and the interval I.

This result follows from the existence of a solution for (3) for λ lying in I and O $(\lambda \neq 0)$

$$\begin{aligned}\psi(\lambda, x) = {} & [y_2'(\lambda, 0)y_3(\lambda, 0) - y_3'(\lambda, 0)\, y_2(\lambda, 0)]y_1(\lambda, x) + \\ & [y_1(\lambda, 0)\, y_3'(\lambda, 0) - y_1'(\lambda, 0)y_3(\lambda, 0)]y_2(\lambda, x) + \\ & [y_1'(\lambda, 0)y_2(\lambda, 0) - y_1(\lambda, 0)y_2'(\lambda, 0)]y_3(\lambda, x)\end{aligned}$$

that satisfies conditions (4) and has the following asymptotic form as $x \to \infty$:

$$\psi(\lambda, x) = a_1e^{-|\beta|x} + a_2e^{i|\beta|x} + a_3e^{-i|\beta|x} + o(1)$$

However, without additional conditions it is obviously impossible to exclude the possibility of the eixstence of eigenvalues "embedded in the continuous spectrum" as we know (/2/, say) for other spectral problems. Such eigenvalues are found at points of I and O where

$$y_i(\lambda, 0) = y_i'(\lambda, 0) = 0 \tag{11}$$

Here $y_i(\lambda, x)$ is understood to be a branch of the function $y(\lambda, x)$ decreasing for given λ as $x \to \infty$.

The question of the existence of a spectrum in the right-hand half-plane is of interest in connection with the question of the stability of the original problem (1) and (2). According to Lemma 2, it follows that there is no continuous spectrum in the right-hand half-plane.

Theorem 1. If conditions 1) and 2) are satisfied and

$$\int_0^\infty g_+(x)\,x\,dx < \ln 2, \quad g_+(x) = \begin{cases} g(x), & g(x) \geqslant 0 \\ 0, & g(x) < 0 \end{cases} \tag{12}$$

then there are no eigenvalues of problem (3) and (4) in the right-hand half-plane.

Proof. We will first prove that the eigenvalues in the right-hand half-plane (if they exist) are real. We multiply (3), written for the eigenfunction $\psi_i(\lambda_j, x)$ (λ_j is a certain eigenvalue), by $(1 + \alpha\lambda_j)\overline{\psi_i(\lambda_j, x)}$ and integrate between $x = 0$ and $x = \infty$. We obtain

$$(1 + \alpha \operatorname{Re} \lambda_j)I_i^{(1)} + \operatorname{Re} \lambda_j I_i^{(2)} + [(\operatorname{Re} \lambda_j)^2 - (\operatorname{Im} \lambda_j)^2]I_i^{(3)} = I_i^{(4)} \tag{13}$$

$$\alpha \operatorname{Im} \lambda_j I_i^{(1)} + \operatorname{Im} \lambda_j I_i^{(2)} + 2 \operatorname{Im} \lambda_j \operatorname{Re} \lambda_j I_i^{(3)} = 0 \tag{14}$$

$$I_i^{(1)} = \int_0^\infty |\psi_i''(\lambda_j, x)|^2\,dx, \quad I_i^{(2)} = \int_0^\infty k(x)\,|\psi_i(\lambda_j, x)|^2\,dx$$

$$I_i^{(3)} = \int_0^\infty |\psi_i(\lambda_j, x)|^2\,dx, \quad I_i^{(4)} = \int_0^\infty g(x)\,|\psi_i'(\lambda_j, x)|^2\,dx$$

For $\mathrm{Re}\,\lambda_j \geqslant 0$ it follows from (14) that $\mathrm{Im}\,\lambda_j = 0$.

According to the results in /3/, the number of eigenvalues of problem (3) and (4) in the right-hand half-plane equals* (*Pivovarchik V.N., On the spectrum of quadratic bundles of unbounded operators associated with stability problems. Dep. UkrNIINTI January 13, 1988, 217-UK88, Kiev, 1988.) the number of eigenvalues in the right-hand half-plane for this problem for

$$k(x) \equiv 0,\ \eta = 1 \ \text{ and } \ \alpha = 0.$$

According to Poincare's theorem, the solutions $y_1(\lambda, \eta, x)$ and $y_2(\lambda, \eta, x)$ are entire functions of η for fixed x and $\lambda \neq -\alpha^{-1}$ (η does not occur the boundary condition (10)). Analogously, the derivatives $y_1'(\lambda, \eta, x)$, $y_2'(\lambda, \eta, x)$, that are solutions of (3) differentiated with respect to x for $k(x) \equiv 0$ and the boundary conditions

$$\lim_{x\to\infty} y_1'(\lambda, \eta, x)\, e^{|\beta|x} = -|\beta|, \quad \lim_{x\to\infty} y_2'(\lambda, \eta, x)\, e^{i|\beta|x} = -i|\beta|$$

are entire functions of η. Therefore, the solution $\psi_1(\lambda, \eta, x)$ is also an entire function of η. Moreover, according to Lemma 1 the function $\psi_1(\lambda, \eta, 0)$ is analytic in $\lambda\ (\lambda \neq -\alpha^{-1})$ for fixed η.

According to the theorem on implicit functions given by the analytic equation (/3/ p.473) the zeros $\lambda_j(\eta)$ of the function $\psi_1(\lambda, \eta, 0)$ in the right-hand half-plane are piecewise-analytic in η. Since the zeros of this function are simple in the right-hand half-plane (no collisions occur in the right-hand half-plane), they are analytic in η. For $\eta = 0$ there are no eigenvalues in the right-hand half-plane. The negative part of $g(x)$ obviously does not increase the number of eigenvalues in the right-hand half-plane (it shifts the eigenvalues in the right-hand half-plane to the left).

We set $g(x) = \eta g_+(x)$ in (3) for $k(x) \equiv 0$. For the solution belonging together with the derivative to $L_2(0, \infty)$, this equation is equivalent to the equation

$$y(\lambda, \eta, x) = y_0(\lambda, x) - \eta \int_x^\infty \frac{\sin\beta(x-t)\,dt}{\beta} \int_t^\infty \mathrm{sh}\,\beta(t-s)\,[g_+(s)\,y'(\lambda, \eta, s)]'\,ds \tag{15}$$

where $y_0(\lambda, x)$ is the solution of (3) from $L_2(0, \infty)$ for $k(x) \equiv 0$, $\eta = 0$, and is β one of the values of $\beta(\lambda)$.

After differentiation with respect to x and integration by parts on the right-hand side, (15) is reduced to the form

$$y'(\lambda, \eta, x) = y_0'(\lambda, x) - \eta \int_x^\infty \cos\beta(x-t)\,dt \int_t^\infty \mathrm{ch}\,\beta(t-s)\,g_+(s)\,y'(\lambda, \eta, s)\,ds \tag{16}$$

The solution of (15) can be represented in the form of the series

$$y'(\lambda, \eta, x) = \sum_{n=0}^\infty z_n(\lambda, \eta, x); \quad z_0(\lambda, \eta, x) = y_0'(\lambda, x)$$

$$z_{n+1}(\lambda, \eta, x) = -\eta \int_x^\infty \cos\beta(x-t)\,dt \int_t^\infty \mathrm{ch}\,\beta(t-s)\,g_+(s)\,z_n(\lambda, \eta, s)\,ds$$

Using the obvious inequalities

$$|\cos\beta(x-t)| \leqslant {}^1/_2\,(e^{|\beta|s} + 1), \quad |\mathrm{ch}\,\beta(x-t)| \leqslant {}^1/_2\,(e^{|\beta|t} + 1)$$

that hold for $0 \leqslant x \leqslant t \leqslant s$, we obtain

$$|z_{n+1}(\lambda, \eta, x)| \leqslant |\eta| \left|\int_x^\infty (t-x)\left[\frac{\exp(|\beta|t) + 1}{2}\right]^2 g_+(t)\,|z_n(\lambda, \eta, t)|\,dt\right.$$

whence

$$\begin{gathered} |y'(\lambda, \eta, x)| \leqslant |y_0'(\lambda, x)| \exp \int_x^\infty \Phi(\lambda, \eta, t)\,dt \\ |y'(\lambda, \eta, x) - y_0'(\lambda, x)| \leqslant |y_0'(\lambda, x)| \left[\exp \int_x^\infty \Phi(\lambda, \eta, t)\,dt - 1\right] \\ \Phi(\lambda, \eta, t) = {}^1/_4\, t\eta g_+(t)\,[e^{|\beta|t} + 1]^2 \end{gathered} \tag{17}$$

Inequality (12) is equivalent to the inequality

$$\int_0^\infty \Phi(0, 1, t)\, dt < \ln 2$$

Consequently, for positive sufficiently small λ the inequality

$$\int_0^\infty \Phi(\lambda, 1, t)\, dt < \ln 2$$

is satisfied, from which the impossibility of the equality $y'(\lambda, \eta, 0) = 0$ follows for $\eta \in [0, 1]$ and sufficient by small $\lambda > 0$ when (17) is taken into account.

Therefore as $\eta \in [0, 1]$ increases and for the conditions of the theorem, the eigenvalues in the right-hand half-plane do not originate from the point $\lambda = 0$. The inequality

$$(1 + \alpha \operatorname{Re} \lambda_j) I_i^{(1)} + \operatorname{Re}\lambda_j I_i^{(2)} + (\operatorname{Re} \lambda_j)^2 I_i^{(3)} \leqslant {}^1/_2 g_{\max} [I_i^{(1)} + I_i^{(3)}]$$

follows from (13) for $\operatorname{Im} \lambda_j = 0$.

This last inequality is not satisfied for

$$\operatorname{Re} \lambda_j > \max \left\{ \frac{g_{\max}}{2\alpha} - \frac{1}{\alpha},\ \sqrt{\frac{g_{\max}}{2}} \right\}$$

Therefore, the spectrum in the right-hand half-plane lies in a finite domain, i.e., as η increases from 0 to 1 the eigenvalues are not incident in the right-hand half-plane from infinity. The theorem is proved.

Let us consider a discrete spectrum in the left-hand half-plane.

Theorem 2.1°. If conditions 1) and 2) are satisfied, $g(x) \geqslant 0$ and inequality (12) hold, then a discrete spectrum of the problem (3) and (4) is possible on the half-axis $(-\infty, 0]$ and within the circle O.

2°. If condition 1) is satisfied, $g(x) \equiv 0$, $k(x) \leqslant k_{\max} < \alpha^{-1}$, then a discrete spectrum of problem (3) and (4) is possible on the half-axis $(-\infty, 0]$ and between concentric circles of radii $r_1 = \alpha^{-1}$ and $r_2 = \alpha^{-1}\sqrt{1 - \alpha k_{\max}}$ with centre at the point $\lambda = -\alpha^{-1}$.

3°. If condition 2) is satisfied, $g(x) \leqslant 0$, $k(x) \equiv 0$, then a discrete spectrum of problem (3) and (4) is possible on the half-axis $(-\infty, 0]$ and between circles of radius r_1 with centre at the point $\lambda = -\alpha^{-1}$ and radius $r_3 = \sqrt{\gamma^2 + g_{\max}/(2\alpha)}$ with centre at the point $\lambda = -\gamma$ in the left-hand half-plane, where $\gamma = (4\alpha + g_{\max})/(4\alpha^2)$.

Proof. We multiply (3) written for the eigenfunction $\psi_i(\lambda_j, x)$ by $\overline{\psi_i(\lambda_j, x)}$ and we subtract the complex conjugate equation to (3) multiplied by $\psi_i(\lambda_j, x)$ from the result. After integration with respect to x we obtain that either $\operatorname{Im}\lambda_j = 0$ or

$$I_i^{(2)} + \{\alpha [(\operatorname{Re}\lambda_j)^2 + (\operatorname{Im} \lambda_j)^2] + 2\operatorname{Re}\lambda_j\} I_i^{(3)} + I_i^{(1)} = 0 \qquad (18)$$

If the result of Theorem 1 is taken into account then all assertions of Theorem 2 follow from this formula, except the boundedness of the discrete spectrum by the circle of radius r_3 in the last case.

To prove the last assertion, we obtain from (14) for $\operatorname{Im}\lambda_j \neq 0$, $k(x) \equiv 0$

$$I_i^{(1)} = -\alpha^{-1}\operatorname{Re}\lambda_j I_i^{(3)}$$

It follows from (18) for $k(x) \equiv 0$, $g(x) < 0$ that

$$\{\alpha [(\operatorname{Re} \lambda_j)^2 + (\operatorname{Im} \lambda_j)^2] + 2 \operatorname{Re} \lambda_j\} I_i^{(3)} = \int_0^\infty g(x) \psi_i''(\lambda_j, x) \overline{\psi_i(\lambda_j, x)}\, dx \leqslant \frac{1}{2} \int_0^\infty |g(x)| [|\psi_i''(\lambda_j, x)|^2 + |\psi_i(\lambda_j, x)|^2]\, dx \leqslant \frac{1}{2} g_{\max} \left(1 - \frac{\operatorname{Re}\lambda_j}{\alpha}\right) I_i^{(3)}$$

whence

$$(\operatorname{Re}\lambda_j + \gamma)^2 + (\operatorname{Im}\lambda_j)^2 \leqslant \gamma^2 + g_{\max}/(2\alpha)$$

The theorem is proved.

In the special case when $k(x) \equiv 0$, $g(x) < 0$, a sufficient condition can be obtained for there to be no complex discrete spectrum.

Theorem 3. If condition 2) is satisified, $\eta g_1(x) \leqslant 0$, $g(x) = \eta g_1(x)$, $k(x) \equiv 0$ and the following inequalities are satisfied

$$\alpha > 0, \quad 2 + \sqrt{2} > \exp\left\{\eta \int_0^\infty (A_1 + B_2)\, dt\right\} + \exp\left\{\eta \int_0^\infty (A_2 + B_1) dt\right\} \qquad (19)$$

$$A_1 = {}^3/_2 \mid g_1(t) \mid t + {}^1/_2 \mid g_1'(t) \mid t^2, \quad A_2 = {}^1/_2 \mid g_1(t) \mid t\,(2 + \exp\sqrt{2/\alpha t}) + {}^1/_4 \mid g_1'(t) \mid t^2 (1 + \exp\sqrt{2/\alpha t})$$

$$B_1 = \mid g_1(t) \mid t, \quad B_2 = {}^1/_2 \mid g_1(t) \mid t\,(1 + \exp\sqrt{2/\alpha t})$$

then a discrete spectrum of problem (3) and (4) is possible for $k(x) \equiv 0$ only in the interval $(-2\alpha^{-1}, -\alpha^{-1})$.

Proof. According to the third assertion of Theorem 2, a discrete spectrum is possible in the case under consideration only on the half-axis $(-\infty, 0]$ and between circles of radius r_1 with centre at the point $\lambda = -\alpha^{-1}$ and radius r_3 with centre at the point $\lambda = -\gamma$ in the left-hand half-plane. But it follows from (13) for $\operatorname{Im}\lambda_j = 0$, $k(x) \equiv 0$, $-\alpha^{-1} < \operatorname{Re}\lambda_j \leqslant 0$ that

$$\int_0^\infty g(x) \mid \psi_i(\lambda_j, x) \mid^2 dx > 0$$

but this is impossible for $g(x) \leqslant 0$.

Therefore, it remains to prove that there are no eigenvalues between the circles mentioned. The eigenvalues between the circles, including also the real ones, correspond to zeros of the function $\psi_1(\lambda, \eta, 0)$ which is analytic in λ (for fixed η) outside the circle O with a slit from $\lambda = -2\alpha^{-1}$ to $\lambda = -\infty$, and the entire function η (for fixed λ lying on and outside the circle O). This latter follows from the reasoning presented in the proof of Theorem 1.

Because of the continuity of the zeros of the function $\psi_1(\lambda, \eta, 0)$ in η outside the circle and their absence for $\eta = 0$, it remains to prove that there are no zeros of the function $\psi_1(\lambda, \eta, 0)$ in the circle O and in the interval J when inequality (19) is satisfied. Then because of the monotonicity of the right-hand side of inequality (19) in η the zeros of the function $\psi_1(\lambda, \eta, 0)$ do not occur even for smaller η, consequently, $\psi_1(\lambda, \eta, 0)$ does not vanish outside the circle O.

For λ lying on the circle O and $x \leqslant t \leqslant s$ the inequalities

$$e^{|\beta|(x-s)} \operatorname{ch} \mid \beta \mid (t - s) \leqslant 1$$
$$\mid e^{i|\beta|(x-s)} \operatorname{ch} \mid \beta \mid (t - s) \mid \leqslant {}^1/_2\, [e^{|\beta|(s-t)} + 1]$$
$$e^{|\beta|(x-t)} \leqslant 1, \quad \mid e^{-i|\beta|(x-t)} \mid \leqslant 1$$
$$\mid e^{|\beta|(x-s)} \sin \mid \beta \mid (x - t) \operatorname{sh} \mid \beta \mid (t - s) \mid \leqslant {}^1/_2$$
$$\mid e^{i|\beta|(x-s)} \sin \mid \beta \mid (x - t) \operatorname{sh} \mid \beta \mid (t - s) \mid \leqslant {}^1/_2 e^{|\beta|(s-t)}$$
$$\mid 1 + \alpha\lambda \mid = 1, \quad \mid \beta^{-1} \sin \mid \beta \mid (x - t) \mid \leqslant t - x, \quad \mid \sin \mid \beta \mid (x - t) \mid \leqslant 1$$

hold.

Using these inequalities, we obtain

$$\mid y_1(\lambda, \eta, x) - e^{-|\beta|x} \mid \leqslant e^{-|\beta|x} \left[\exp\left(\eta \int_x^\infty A_1\, dt\right) - 1\right] \tag{20}$$

$$\mid y_2(\lambda, \eta, x) - e^{-i|\beta|x} \mid \leqslant \exp\left(\eta \int_x^\infty A_2\, dt\right) - 1$$

in the same way as when deriving (9).

An analogous investigation of the equations

$$y_1'(\lambda, \eta, x) = - \mid \beta \mid e^{-|\beta|x} - \int_x^\infty \cos \mid \beta \mid (x - t) \int_t^\infty \frac{\operatorname{ch} \mid \beta \mid (t - s)}{1 + \alpha\lambda} \eta g_1(s)\, y_1'(\lambda, \eta, s)\, ds$$

$$y_2'(\lambda, \eta, x) = - i \mid \beta \mid e^{-i|\beta|x} - \int_x^\infty \cos \mid \beta \mid (x - t) \int_t^\infty \frac{\operatorname{ch} \mid \beta \mid (t - s)}{1 + \alpha\lambda} \eta g_1(s)\, y_2'(\lambda, \eta, s)\, ds$$

obtained by differentiating (6) with respect to x for $k(x) \equiv 0$ after integration by parts, results in the inequalities

$$\mid y_1'(\lambda, \eta, x) + \mid \beta \mid e^{-|\beta|x} \mid \leqslant \mid \beta \mid e^{-|\beta|x} \left[\exp\left(\eta \int_x^\infty B_1\, dt\right) - 1\right] \tag{21}$$

$$|y_2'(\lambda,\eta,x)+i|\beta|e^{-i|\beta|x}|\leqslant|\beta|\left[\exp\left(\eta\int_x^\infty B_2\,dt\right)-1\right]$$

Using inequalities (20) and (21), we obtain from the definition of $\psi_1(\lambda,\eta,x)$

$$|\psi_1(\lambda,\eta,0)+|\beta|(1-i)|\leqslant|\beta|\left\{\exp\left[\eta\int_0^\infty(A_1+B_2)\,dt\right]+\exp\left[\eta\int_0^\infty(A_2+B_1)\,dt\right]-2\right\}$$

It is seen from the latter formula that $\psi_1(\lambda,\eta,0)\neq 0$ follows from inequality (19). The first inequality of (21) is satisfied not only in the circle O but also in the interval J. Consequently $y_1'(\lambda,\eta,0)\neq 0$, follows from the inequality (19), i.e., (11) is not satisfied, meaning, there are no eigenvalues in this interval. The theorem is proved.

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AXISYMMETRIC FLEXURAL OSCILLATIONS OF A THIN DISC*

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Using methods of the theory of singular perturbations /1-3/, we construct the asymptotic forms of the eigenfrequencies of flexural low-frequency oscillations of a thin disc. Application of the method of homogeneous solutions /4/ or the superposition method /5/ reduces the problem under consideration to an infinite system of linear algebraic equations. Unlike these approaches, the theory of singular perturbations enables us to obtain explicit formulae for corrections to the oscillation eigenfrequencies obtained from the classical theory of plates.

1. Formulation of the problem. We consider the problem of the axially-symmetric flexural oscillations of a thin disc of radius a and thickness $2h$ $(\varepsilon=h/a\ll 1)$ in a system of cylindrical coordinates (r,φ,z). The planes $z=\pm h$ and the side surface $r=a$ are free from stresses.

In dimensionless coordinates $\rho=r/a$, $\xi=z/h$ the problem may be written in the form

$$(1-2\nu)\partial_\xi^2u_r+\varepsilon\partial_\rho\partial_\xi u_z+2(1-\nu)\varepsilon^2\partial_\rho(\rho^{-1}\partial_\rho(\rho u_r))+\mu u_r=0 \tag{1.1}$$
$$2(1-\nu)\partial_\xi^2u_z+\varepsilon\rho^{-1}\partial_\rho(\rho\partial_\xi u_r)+(1-2\nu)\varepsilon^2\Delta u_z+\mu u_z=0$$

$$G(\partial_\xi u_r+\partial_\rho u_z)_{\xi=\pm1}=0 \tag{1.2}$$
$$d[2(1-\nu)\partial_\xi u_z+2\nu\rho^{-1}\partial_\rho(\rho u_r)]_{\xi=\pm1}=0$$

$$d[2(1-\nu)\partial_\rho u_r+2\nu(\partial_\xi u_z+\rho^{-1}u_r)]_{\rho=1}=0 \tag{1.3}$$
$$G(\partial_\xi u_r+\partial_\rho u_z)_{\rho=1}=0$$

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